Exam Lie Groups in Physics

Date	November 9, 2016
Room	5419.0119
Time	9:00 - 12:00
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- Raise your hand for more paper
- You are not allowed to use the lecture notes, nor other notes or books
- The weights of the **four** problems are given below
- Answers may be given in Dutch
- Illegible handwriting will be graded as incorrect
- Good luck!

Weighting

1a)	5	2a)	10	3a)	6	4a)	5
1b)	5	2b)	10	3b)	6	4b)	8
1c)	5	2c)	8	3c)	6	4c)	5
1d)	6						
1e)	5						

$$\text{Result} = \frac{\sum \text{points}}{10} + 1$$

Problem 1

(a) Consider the sets of real numbers R and positive real numbers R^+. Indicate the composition laws under which these sets form Lie groups? Explain your answers.

(b) Show that $\mathsf{R} \cong \mathsf{R}^+$.

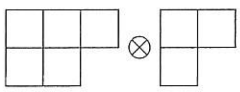
(c) Show that $\mathsf{R}/\mathsf{Z} \cong U(1)$, where Z denotes the group of integers and $U(1)$ the group of unitary 1×1 matrices.

(d) Consider the group $(\{e^{i\theta/2} | 0 \leq \theta \leq 4\pi\}; \times)$. Show whether this group is isomorphic to $U(1)$ or not. If not, show how it is related.

(e) Give an example from physics where $U(1)$ plays a role as an exact or approximate symmetry.

Problem 2

(a) Decompose the following direct product of irreps of the Lie algebra $su(n)$



into a direct sum of irreps of $su(n)$, in other words, determine its Clebsch-Gordan series.

(b) Write down the dimensions of the irreps appearing in the obtained decomposition for $su(2)$ and $su(3)$. Indicate complex conjugate irreps whenever appropriate.

(c) Relate the decomposition for $su(2)$ to the corresponding case of addition of angular momenta in Quantum Mechanics.

Problem 3

Consider the Lie group $U(n)$ of unitary $n \times n$ matrices and its subgroup $SU(n)$ of unitary $n \times n$ matrices with determinant equal to 1.

(a) Write down the properties of the Lie algebras of $U(n)$ and $SU(n)$, including their dimensions.

Consider the Lie group $O(n)$ of orthogonal $n \times n$ matrices and its subgroup $SO(n)$ of orthogonal $n \times n$ matrices with determinant equal to 1.

(b) Write down the properties of the Lie algebras of $O(n)$ and $SO(n)$, including their dimensions.

(c) Explain why the dimensions for $O(n)$ and $SO(n)$ coincide, whereas for $U(n)$ and $SU(n)$ they do not.

Problem 4

Consider the group of Lorentz transformations $L^{\mu}{}_{\nu}$.

(a) Demonstrate that invariance of the Minkowski metric under Lorentz transformations implies that $(L^0{}_0)^2 \geq 1$.

Consider the Lorentz algebra given by

$$[J^j, J^k] = i\epsilon_{jkl} J^l, \quad [J^j, K^k] = i\epsilon_{jkl} K^l, \quad [K^j, K^k] = -i\epsilon_{jkl} J^l.$$

(b) Show by explicit calculation that the following two operators are Casimir operators of the Lorentz group:

$$C_1 = \vec{J}^2 - \vec{K}^2, \quad C_2 = 2\vec{J} \cdot \vec{K}$$

(c) Explain the use of Casimir operators.